Test of universal finite-size scaling in two-dimensional site percolation

1997 J. Phys. A: Math. Gen. 30 L301

(http://iopscience.iop.org/0305-4470/30/10/001)

J. Phys. A: Math. Gen. **30** (1997) L301–L306. Printed in the UK PII: S0305-4470(97)81102-X

LETTER TO THE EDITOR

Test of universal finite-size scaling in two-dimensional site percolation

A Aharony and D Stauffer†

School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

Received 20 January 1997, in final form 19 February 1997

Abstract. Traditional two-scale-factor universality, concerning the number of clusters within a correlation volume near the percolation threshold, is reconfirmed by a comparison of site percolation on square and triangular lattices. We also test universality *at* the threshold, e.g. of the ratio ξ/L, where ξ is the correlation length and L the lattice size. At the threshold, the result is sensitive to the system's shape and aspect ratio, boundary conditions, the algorithm, the detailed units measuring ξ and L, and possibly other factors as yet unexplored.

Near second-order phase transitions, it is not only the critical exponents which are 'universal', i.e. independent of lattice type and of many other short-range details for a given dimensionality, but also certain amplitude combinations [1]. Each such combination corresponds to a scaling law relating the corresponding critical exponents. For example, two-scale-factor universality concerns the universality of the singular part of the free energy within the correlation volume, in units of $k_B T_c$. This corresponds to the hyperscaling law $2 - \alpha = d\nu$, where α and ν describe the divergence of the specific heat and the correlation length, respectively, and d is the dimensionality. Less clear is the application of this universality concept to *finite* samples. As the lattice size L decreases below the bulk correlation length, singularities become cut off and replaced by powers of L (see later). Here we address the following questions: Does the crossover, where finite-size effects become clearly noticeable, happen at a universal ratio of crossover length $L_\times$ to the correlation length ξ? Does it depend on the quantity under study? Does it depend on the detailed way by which the relevant quantity is measured? In addition, we study the singular L-dependence of various quantities *at* the transition point, and discuss possible universal relations among them. We study these problems for two-dimensional site percolation, where each of the $L \times L$ sites of a large lattice is randomly occupied with probability p.

Translating the work of Privman and Fisher [2] from the thermal case to percolation, and combining it with earlier results on universal amplitude ratios away from the percolation threshold p_c [3], we assume that the singular part in the cluster number density within a d-dimensional cube of linear size L obeys the scaling form

$$f^{(s)} = L^{-d} Y(C_1(p - p_c)L^{1/\nu}, C_2 h L^{\Delta/\nu}) \tag{1}$$

where h is the ghost field, ν and $\Delta = \beta + \gamma$ are the critical exponents for connectivity length and field, respectively, and Y is a universal function after an appropriate choice of the

† Present and permanent address: Institute for Theoretical Physics, Cologne University, 50923 Köln, Germany.

system dependent non-universal scale factors C_1 and C_2. This theory immediately predicts that at p_c one has $f^{(s)} = f_c/L^d$, with the *universal* amplitude $f_c = Y(0,0)$. Indeed, the universality of the amplitude f_c was recently observed numerically (and given some heuristic justification) by Ziff *et al* [4], for specific aspect ratios and boundary conditions. However, they did not connect their findings to the above general scaling scheme. A major issue in the present letter concerns the dependence of the function $Y(x, y)$ on its first argument, $x = C_1(p - p_c)L^{1/\nu}$. One expects that when x becomes large and negative (or positive) the results should no longer depend on L, and thus $Y(x, 0)$ should behave as $C_{f,\pm}|x|^{d\nu}$, with the universal coefficients $C_{f,+}$ or $C_{f,-}$ above and below p_c, yielding $f^{(s)} = D_\pm|p - p_c|^{2-\alpha}$, with $2 - \alpha = \gamma + 2\beta = d\nu$ and $D_\pm = C_{f,\pm}C_1^{d\nu}$. The crossover between the two asymptotic forms of $f^{(s)}$ should occur at some intermediate universal value of x, which for $x < 0$ we may call $-x_\times$. A possible definition of this value could be where $Y(x, 0)$ deviates by a certain fraction (say 40%) from its asymptotic form $C_{f,\pm}|x|^{d\nu}$. This then defines a crossover length, $L_\times = (x_\times/[C_1(p_c - p)])^\nu$. An interesting question concerns the relation of this length to the percolation correlation length, which is defined (apart from a factor $\sqrt{2}$ [5–7]) as the root-mean-square radius of gyration of the finite clusters, which diverges as $\xi = \xi_0(p_c - p)^{-\nu}$. Since both lengths are expected to diverge with the *same* exponent ν, one is tempted to expect that the ratio $L_\times/\xi = (x_\times/C_1)^\nu/\xi_0$ is universal. Indeed, the theory of Privman and Fisher [2] also assumes that ξ should obey the scaling relation

$$\xi = LX(C_1(p - p_c)L^{1/\nu}, C_2hL^{\Delta/\nu}) \tag{2}$$

with the universal function $X(x, y)$ and without any additional scale factors. The universal function depends on shape, boundary condition, and surface fields. Using similar arguments as above, one concludes that for $p < p_c$ and large L one has the universal behaviour $X(x, 0) = C_{\xi,-}|x|^{-\nu}$, hence $\xi_0 = C_{\xi,-}C_1^{-\nu}$, so that indeed $L_\times/\xi$ is universal. Equation (2) also predicts that *at* p_c one has $\xi = LX(0, 0)$, with the universal amplitude $\xi_c = X(0, 0)$. This universality is one of the issues discussed in the present letter. Given the universality of $L_\times/\xi$, one can now test universality by using either ξ or any crossover length coming from any physical quantity. For example, Kapitulnik *et al* [8] identified a crossover length from P_∞. *A priori*, there is no reason that different quantities should not exhibit crossover at different sizes. The point we emphasize is the universal ratios among these lengths.

Equation (1) immediately yields expressions for the fraction of sites in the infinite spanning cluster, $P_\infty = \partial f^{(s)}/\partial h$ and the mean cluster size $S = \partial^2 f^{(s)}/\partial h^2$: $P_\infty = C_2L^{-\beta/\nu}Y'(x, y)$ and $S = C_2^2L^{\gamma/\nu}Y''(x, y)$. These imply that at p_c one has universal ratios $L^{-d}S/P_\infty^2$ and (using equation (2)) $\xi^{-d}S/P_\infty^2$. Below p_c and for large L one has $S = C(p_c - p)^{-\gamma}$ and $f^{(s)} = D(p_c - p)^{2-\alpha}, D = D_-$, while above p_c one has $P_\infty = B(p - p_c)^\beta$. The above theories also relate the amplitudes B and C to the scale factors C_1 and C_2 and to universal coefficients arising from $Y(x, y)$, and these result in universality of any amplitude combination in which C_1 and C_2 cancel. Specifically, this yields, for example, the universality of the ratios DC/B^2 and $D\xi_0^2$ [3].

The literature contains several different normalizations of critical quantities: per site, per occupied site, per unit area. To avoid confusion on this, we have re-evaluated several such quantities by standard Monte Carlo methods [5]. Specifically, we used the Leath (up to $L = 2001$) and the Hoshen–Kopelman (up to $L = 10^5$) algorithms [9]. For Leath, one cluster at a time starts to grow from the centre of a lattice, while for Hoshen–Kopelman all clusters formed by filling the whole lattice are randomly counted. Mostly, we used Leath for S and ξ and Hoshen–Kopelman for P_∞; both programs gave the same results for S and P_∞ away from p_c. Comparison at p_c is more complex, as discussed below.

In order to obtain two-scale-factor universality for $f^{(s)}\xi^d$, if the singular 'free energy' $f^{(s)}$ is measured per site then the correlation volume ξ^d should be multiplied by the density of sites. This can be achieved if ξ is measured not in units of the nearest-neighbour distance but in units of the square root of the area per site. In the square lattice this distinction does not matter, but in the triangular lattice then ξ_0 is larger by a factor $(2/\sqrt{3})^{1/2}$ than if measured in units of the nearest-neighbour distance. We used the normalization of ξ^2 by the area per site. (Also S and P_∞ are measured as quantities per site and not per occupied site.)

We start by checking universality away from p_c. From our numerical simulations we find B, C, and ξ_0 to be 0.78, 0.072, and 0.52 on the triangular (TR) and 0.91, 0.102, and 0.52 on the square (SQ) lattice, with accuracy of about 1%. (Here we used the two-dimensional known exponents $\alpha = -2/3$, $\beta = 5/36$, $\gamma = 43/18$, $\nu = 4/3$ [5].) Domb and Pearce [10] found $D = -4.37$ for the triangular lattice and -4.24 for square bond percolation. Assuming two-scale-factor universality, and noting the fact that series [11] give the same value $\xi_0 = 0.52$ for both bond and site square lattice percolation, we assume that $D = -4.24$ also for square site percolation. With these numbers the traditional universal combinations give

$$DC/B^2 = -0.52(\mathrm{TR}) = -0.52(\mathrm{SQ}) \tag{3}$$

$$D\xi_0^2 = -1.18(\mathrm{TR}) = -1.15(\mathrm{SQ}) \tag{4}$$

$$B^2\xi_0^2/C = 2.28(\mathrm{TR}) = 2.20(\mathrm{SQ}). \tag{5}$$

The differences between TR and SQ are thus smaller than our combined error bars, and universality is confirmed. (For square bond percolation, Daboul *et al* [11] found $B^2\xi_0^2/C \simeq 2.21$.)

For finite systems, exactly at $p = p_c$ we have for $L \times L$ sites

$$S = C_c L^{\gamma/\nu} \qquad P_\infty = B_c L^{-\beta/\nu} \qquad \xi = \xi_c L \qquad f^{(s)} = f_c/L^d. \tag{6}$$

We found B_c, C_c and ξ_c to be 0.354, 0.015, and 0.301 for TR and 0.430, 0.023, and 0.314 for SQ, using Leath for S and ξ and Hoshen–Kopelman for P_∞, with errors of the order of 10^{-3}. As stated, theory predicts that ξ_c should be universal. Figure 1 shows our data in the form of ξ/L, where L and ξ are given in units of the square root of the area per site, which was found in equations (4) and (5) to be appropriate. L can be defined as the square root of the lattice area (as used for (6)) or as the side length of the lattice, without changing our conclusions. At least for $L > 20$, the data for the square lattice are consistently above those for the triangular one, and the extrapolated values seem to differ by at least 4%. One way to explain these differences would be to identify the SQ and TR lattices with different universality classes. In fact, Ziff *et al* [4] emphasize that the $L \times L$ triangular lattice has an aspect ratio of $\sqrt{3}/2$, and thus need not belong to the same universality class as the square lattice, with an aspect ratio of 1. Indeed, their values of f_c for these two lattices differ by about 0.5%. In addition, possible systematic errors (e.g. with the random number generators) make it desirable to have independent checks on our calculations, in particular of figure 1. Our amplitude $\xi_0 = 0.52$ differs from the $\xi_0 = 0.63$ for TR of Corsten *et al* [7], even though we used roughly their method. (We confirmed, however, their controversial [12] conclusion that the connectivity length above p_c is about four times, and not two times, smaller than below p_c at the same small distance from p_c.)

As mentioned after equation (2), a further check of universality at p_c concerns the ratios $L^{-d}S/P_\infty^2$ and $\xi^{-d}S/P_\infty^2$. Using our results, we find

$$B_c^2/C_c = 8.4(\mathrm{TR}) = 8.0(\mathrm{SQ}) \tag{7}$$

$$B_c^2\xi_c^2/C_c = 0.76(\mathrm{TR}) = 0.79(\mathrm{SQ}). \tag{8}$$

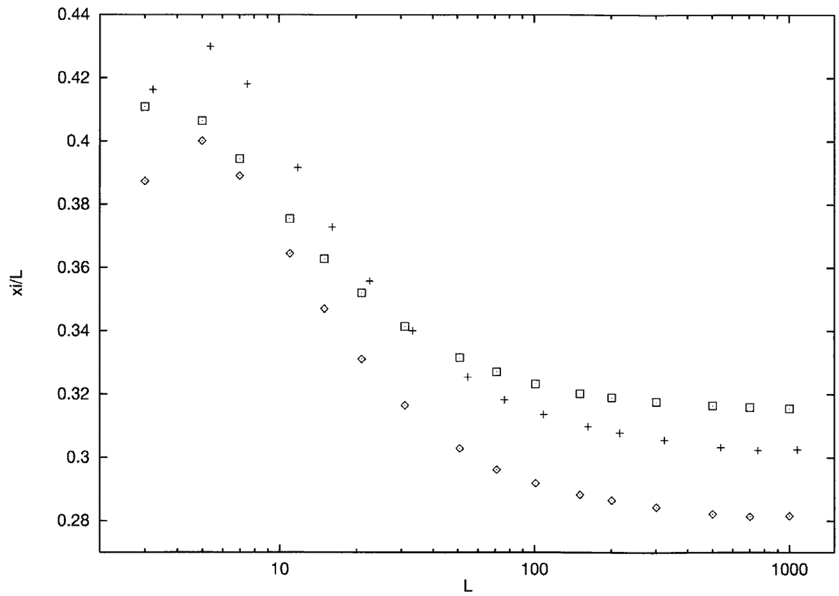

Figure 1. Ratio of connectivity length (defined via the cluster radii), to system length L at the percolation threshold for nearly 10^7 SQ (squares, $p_c = 0.592\,746$ [14]) and TR (+, diamonds, $p_c = 1/2$) lattices. Both lengths are normalized in units of the square root of the area per site; their ratio does not change if they are measured in units of the nearest-neighbour distance. For TR, we show here L defined both as side length of the lattice (+) and as (area)$^{1/2}$ (diamonds).

The two numbers in equation (7) are within the errors of each other. We emphasize again that equations (7) and (8) used C_c from the Leath algorithm, which stops if the cluster which started in the lattice centre touches the upper or lower boundary in which case this cluster is ignored in our averages; periodic boundary conditions were applied horizontally. The amplitudes C_c and ξ_c refer to this algorithm and B_c to Hoshen–Kopelman. When we used the Hoshen–Kopelman algorithm, with free boundaries, counting all clusters even if they touch the boundaries, the mean cluster size in both lattices has a ten times bigger amplitude C_c at p_c than in the Leath algorithm. This surprising ratio arises due to the fact that in the Leath algorithm we do not include many of the largest clusters (including the spanning one, if it exists). Figure 2 shows $P_\infty^2 L^2/S$ at $p = p_c$, and with *all* quantities taken from the Hoshen–Kopelman algorithm; now the two lattices give ratios which differ by about 2%, and this is not likely to be a reflection of our statistical errors (we estimate the error in each point by 0.1%). Could this be related to different aspect ratios? In any case, we draw attention to the important fact that universal ratios *at* p_c depend strongly on the algorithm, even if their corresponding counterparts away from p_c do not.

It has been noticed before that sometimes universality may depend on a variety of additional parameters, especially exactly at the percolation threshold. For example, the probability of finding a spanning cluster depends on the aspect ratio [13], the spanning rule [14, 15] and the boundary conditions [15, 16], although it remains universal when switching from site to bond percolation [17] or from square to triangular lattice [15]. Thus all our amplitudes in equation (6) may depend on these details (we checked this explicitly for the aspect ratio); a square lattice with aspect ratio equal to that of our triangular lattice and tilted by 45° converges nearly to the middle curve in figure 1 for the triangular lattice. At

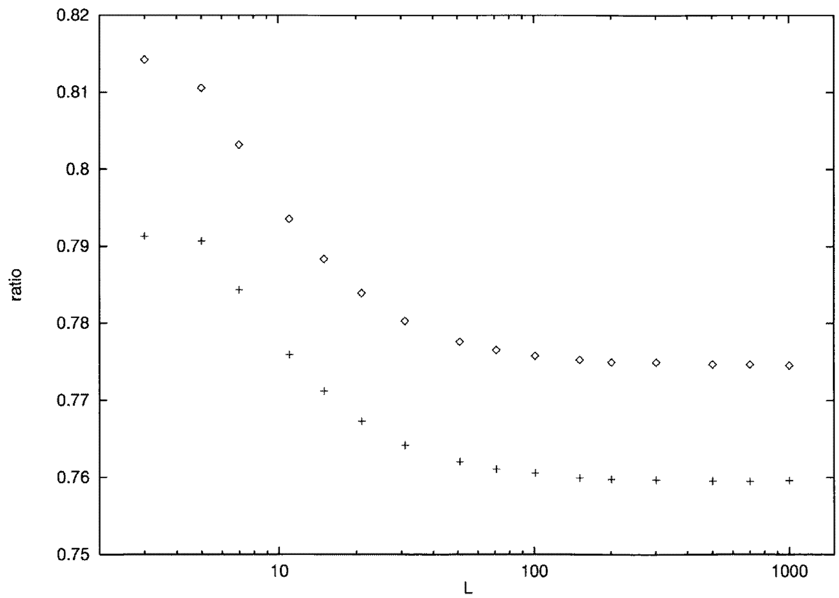

Figure 2. Ratio $P_\infty^2 L^2/S$ at the percolation threshold for nearly 10^7 square (top) and triangular (bottom) lattices. All quantities were obtained using the Hoshen–Kopelman algorithm.

present, the concepts of aspect ratio and of the 'size' L do not seem to be sufficient for uniquely describing the lattice. In the present paper we talked about an $L \times L$ lattice, which was sheared by an angle of 60° to become triangular. Our results lead us to suspect that, in general, universality may depend on both this angle *and* the ratio M/L for an $M \times L$ system. We hope that this paper will stimulate discussions of this point.

In conclusion, although two-scale-factor universality does hold near the percolation threshold, the statement that ξ/L is universal *at* the threshold is not obvious, and requires a careful discussion of the units of both ξ and L, as well as the dependence on the shape of the sample and other details. Similar questions arise concerning amplitude combinations such as in equation (7). It would be nice to extend our results to other lattices, of other shapes and aspect ratios, in order to clarify the correct rules for such units and for universality. It would also be nice to have more extended tests of the universality of the whole function $Y(x, y)$, and connect the universal quantities found here with the amplitude f_c measured by Ziff *et al*. Many of our conclusions also apply to other critical points, and it would be nice to investigate similar questions for other cases, e.g. for the Ising model.

We thank R M Ziff, V Privman and N Jan for helpful discussions, the German Israeli Foundation and the SFB 341 for support, and K F A Jülich for time on Cray-T3E.

References

[1] Privman V, Hohenberg P C and Aharony A 1991 *Phase Transitions and Critical Phenomena* vol 14, ed C Domb and J L Lebowitz (London: Academic) p 1

[2] Privman V and Fisher M E 1984 *Phys. Rev.* B **30** 322
Privman V (ed) 1990 *Finite Size Scaling and Numerical Simulation of Statistical Systems* (Singapore: World Scientific) p 18

[3] Aharony A 1980 *Phys. Rev.* B **22** 400

[4] Ziff R M, Finch S R and Adamchik V 1997 *Preprint*
[5] Stauffer D and Aharony A 1994 *Introduction to Percolation Theory* (London: Taylor and Francis)
[6] Essam J W 1980 *Rep. Prog. Phys.* **43** 833
[7] Corsten M, Jan N and Jerrard R 1989 *Physica* A **156** 781
[8] Kapitulnik A, Aharony A, Deutscher G and Stauffer D 1984 *J. Phys. A: Math. Gen.* **16** L269
[9] Leath P L 1976 *Phys. Rev.* B **14** 5046
Hoshen J and Kopelman R 1976 *Phys. Rev.* B **14** 3438
[10] Domb C and Pearce C J 1976 *J. Phys. A: Math. Gen.* **9** L137
[11] Daboul D *et al* 1997 unpublished
[12] Chayes J T, Chayes L, Grimmett G R, Kesten H and Schonmann R H 1989 *Ann. Prob.* **17** 1277
[13] Vicsek T and Kertész J 1981 *Phys. Lett.* **81A** 51
[14] Ziff R M 1992 *Phys. Rev. Lett.* **69** 2670
[15] Hovi J P and Aharony A 1996 *Phys. Rev.* E **53** 235
[16] Gropengiesser U and Stauffer D 1994 *Physica* A **210** 320
Hu C K 1994 *J. Phys. A: Math. Gen.* **27** L813
[17] Stauffer D, Adler J and Aharony A 1994 *J. Phys. A: Math. Gen.* **27** L475